PRINCIPLES OF ANALYSIS
LECTURE 9 - ARITHMETIC OF SEQUENCES

PAUL L. BAILEY

1. Definitions of Sup and Inf

Recall the following definitions.
Let $S \subset \mathbb{R}$ and let $x \in \mathbb{R}$.
We say that $x = \max S$ if

(a) $x \geq s$ for every $s \in S$;
(b) $x \in S$.

We say that $x = \min S$ if

(a) $x \leq s$ for every $s \in S$;
(b) $x \in S$.

We say that $x = \sup S$ if

(a) $x \geq s$ for every $s \in S$;
(b) $a \geq s$ for every $s \in S \Rightarrow a \geq s$.

We say that $x = \sup S$ if

(a) $x \geq s$ for every $s \in S$;
(b) $a \geq s$ for every $s \in S \Rightarrow a \geq s$.

2. Examples of Sup and Inf

Example 1. Let S be a nonempty bounded subsets of $\mathbb{R}$. Show that $\inf S \leq \sup S$. What can be said if $\inf S = \sup S$?

Proof. Since S is nonempty, there exists $s \in S$. Then $\inf S \leq s$ and $s \leq \sup S$. By transitivity of order, $\inf S \leq \sup S$.

If $\inf S = \sup S$, then S contains only one element. □

Example 2. Let S and T be nonempty bounded subsets of $\mathbb{R}$. Show if $S \subset T$, the $\inf T \leq \inf S \leq \sup S \leq \sup T$.

Proof. Let $s \in S$. Then $s \in T$, so $\inf T \leq s$. Thus $\inf T$ is a lower bound for S, so $\inf T \leq \inf S$. Similarly, $\sup S \leq \sup T$. That $\inf S \leq \sup S$ is true is above. □

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Example 3. Let S and T be nonempty bounded subsets of $\mathbb{R}$. Show that $\sup(S \cup T) = \max\{\sup S, \sup T\}$.

Proof. Either $\max\{\sup S, \sup T\} = \sup S$ or $\max\{\sup S, \sup T\} = \sup T$.

Suppose that $\max\{\sup S, \sup T\} = \sup S$; in this case, $\sup T \leq \sup S$. Since $S \subset S \cup T$, we have $\sup S \leq \sup(S \cup T)$ by part (a).

Now let $x \in S \cup T$. Then x is either in S or T. If $x \in S$, then $x \leq \sup S$. If $x \in T$, then $x \leq \sup T \leq \sup S$. Thus $\sup S$ is an upper bound for $S \cup T$. Therefore $\sup(S \cup T) \leq \sup S$.

Since $\sup S \leq \sup(S \cup T)$ and $\sup(S \cup T) \leq \sup S$, it follows that $\sup S = \sup(S \cup T)$.

Finally, if $\max\{\sup S, \sup T\} = \sup T$, the above proof is valid, with the roles of S and T reversed. □

Example 4. Show that if $a > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

Proof. Let $b = \max\{a, \frac{1}{a}\}$. By the Archimedian property, there exists $n \in \mathbb{N}$ such that $n > b$. Since $a \leq b$, we have $a < n$. Also since $\frac{1}{a} \leq b$, we have $\frac{1}{a} < n$. Thus by Theorem 3.2.(vii), we have $\frac{1}{n} < a$. □

Example 5. Let $a, b \in \mathbb{R}$ such that $a < b$. Show that there exist infinitely many rational numbers between a and b.

Proof. Suppose not. The the set $S = (a, b) \cap \mathbb{Q}$ is finite, so it has a minimum, say $c = \min S$. But then Theorem 4.7 tells us that there exists $d \in \mathbb{Q}$ such that $a < d < c$. But then $d < b$, so $d \in S$. This contradicts that $c = \min S$. □

Example 6. Let A and B be nonempty bounded subsets of $\mathbb{R}$ and let

$$S = \{x \in \mathbb{R} \mid x = a + b \text{ for some } a \in A, b \in B\}.$$

(a) Show that $\sup S = \sup A + \sup B$.
(b) Show that $\inf S = \inf A + \inf B$.

Lemma 1. *Let $A \subset \mathbb{R}$ be bounded above and suppose that $x < \sup A$. Then there exists $a \in A$ such that $x < a$.*

Proof of Lemma. Suppose not; then for every $a \in A$, we have $a \leq x$. Then x is an upper bound for A, so $\sup A \leq x$, contrary to our assumption on x. □

Proof of Example. We prove **(a)**; the proof for **(b)** is symmetric. It suffices to show that $\sup S \leq \sup A + \sup B$ and that $\sup A + \sup B \leq \sup S$.

Let $s \in S$. Then $s = a + b$ for some $a \in A$ and $b \in B$. Then $a \leq \sup A$ and $b \leq \sup B$, so $a + b \leq \sup A + \sup B$. Thus $\sup A + \sup B$ is an upper bound for S, so $\sup S \leq \sup A + \sup B$.

Suppose that $\sup S < \sup A + \sup B$. Then $\sup S - \sup B < \sup A$, so there exists $a \in A$ such that $\sup S - \sup B < a$. From this, $\sup S - a < \sup B$, so there exists $b \in B$ such that $\sup S - a < b$. Let $s = a + b \in S$. We have $\sup S < s$, a contradiction. Therefore $\sup A + \sup B \leq \sup S$. □

3. Arithmetic of Sequences

Lemma 2. *Let $a, b \in \mathbb{R}$. Then $|ab| = |a||b|$.*

Reason. Break this into four cases and see the result. □

Proposition 1. *Let $\{s_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers, and let $k \in \mathbb{R}$. Then*

$$k \lim_{n\to\infty} s_n = \lim_{n\to\infty} (ks_n).$$

Proof. Let $\epsilon > 0$, and set $s = \lim_{n\to\infty} s_n$. Since $s_n \to s$, there exists $N \in \mathbb{Z}^+$ such that

$$|s_n - s| < \frac{\epsilon}{k}.$$

Then

$$|ks_n - ks| < \epsilon.$$

□

Proposition 2. *Let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers. Then the sequence $\{s_n + t_n\}_{n=1}^{\infty}$ converges, and*

$$\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n.$$

Proposition 3. *Let $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ be convergent sequences of real numbers. Then the sequence $\{s_n t_n\}_{n=1}^{\infty}$ converges, and*

$$\lim_{n\to\infty} (s_n t_n) = (\lim_{n\to\infty} s_n)(\lim_{n\to\infty} t_n).$$

Proposition 4. *Let $\{s_n\}_{n=1}^{\infty}$ be a convergent sequence of nonzero real numbers. Then*

$$\frac{1}{\lim_{n\to\infty} s_n} = \lim_{n\to\infty} (\frac{1}{s_n}).$$

Lemma 3. *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of nonzero real numbers such that $\lim_{n\to\infty} |s_n|$ converges to a positive real number. Then there exists $m > 0$ such that $|s_n| > m$ for all n.*

Department of Mathematics and CSCI, Southern Arkansas University
E-mail address: plbailey@saumag.edu